PMM U.S.S.R.,Vol.47,No.1,pp.9-13,1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

UDC 531.36

ON THE LIAPUNOV EXPONENTS OF A LINEAR SYSTEM WITH MARKOV COEFFICIENTS*

S.M. KHRISANOV

Conditions are obtained for the representation of the moments of the solutions of homogeneous linear systems with Markov coefficients, as a matrix-valued exponent. Such a representation is the analog of the Floquet–Liapunov representation for the fundamental matrix of solutions of a homogeneous linear system with periodic coefficients; from it follows the possibility of finding rigorous Liapunov exponents of the system being examined.

1. Let η_t be a vector-valued random process defined on the interval of time $[t_0, \infty)$, for which the vector $m(t)$ of first moments, the matrix $M(t), \ldots$ of second moments, etc. exist for each t from the given interval. The numbers or symbols determined by the formulas

$$\mathrm{X}^{(1)}\eta_t = \lim_{t\to\infty} \frac{1}{t} \ln \| m(t) \|, \quad \mathrm{X}^{(2)}\eta_t = \lim_{t\to\infty} \frac{1}{t} \ln \| M(t) \|, \ldots$$

are called Liapunov exponents in the sense of the moments of corresponding order. For non-random functions these definitions pass naturally to the known ones /1/. Let ξ_t be a uniform Markov process on a measurable phase space $U = \{u\}$ with transition function $P(t, u, \Gamma)$. Let L and L^* be infinitesimal operators corresponding to the semigroups of operators

$$T_t\varphi(u) = \int_U \varphi(y) P(t, u, dy), \quad T_t^* Q(\Gamma) = \int_U P(t, u, \Gamma) Q(du)$$

which act in Banach spaces of measurable bounded functions $\{\varphi(u)\}$ and of finite generalized measures $\{Q(\Gamma)\} = H$ on U. The density of probabilities distribution $p(t, u)$ of process ξ_t can be obtained as the solution of the equation $\partial p/\partial t = L^*p$, $p = p(t, u)$, $p(0, u) = p_0(u)$. We assume that process ξ_t is ergodic and that $q(u)$ is the corresponding unique stationary probabilities distribution with convergence rate estimated by

$$q(u) = \lim_{t\to\infty} p(t, u), \quad |p(t, u) - q(u)| < R \exp(-\lambda t) \tag{1.1}$$

with some constants R and $\lambda > 0$. We consider the linear differential equations system

$$X^{\cdot} = (A + \mu B(\xi_t)) X \tag{1.2}$$

in which X is an n-dimensional vector, $A = (a_{ij})$. $B = (b_{ij}(u))$ are nth-order matrices the first being a constant and the second a measurable function on set U.

Theorem. Let the matrix $B(u)$ be bounded on U and let the eigenvalues $\{\alpha_j, j = 1, \ldots, n\}$ of matrix A admit of the estimates

$$\mathrm{Re}(\alpha_i - \alpha_j) < c = \mathrm{const} < \lambda \tag{1.3}$$

for any $i, j = 1, \ldots, n$. Then under the conditions listed above the mean vector $m(t) = EX(t)$ of the solution of system (1.2) with sufficiently small μ admits of the representation

$$m(t) = \exp(Kt)(C + o)\, m(0), \quad \mathrm{Det}\, C \neq 0 \tag{1.4}$$

where K and C are constant nth-order matrices and o is an infinitesimal matrix as $t \to \infty$.

Proof. We write out the equation for the vector $m(t, u) = E(X(t), \xi_t = u)$ of first partial moments in the form /2/

$$\frac{\partial m(t, u)}{\partial t} = (A + \mu B(u))\, m(t, u) + L^* m(t, u) \tag{1.5}$$

The solutions of this system are connected with the vector of first moments by the formula

*Prikl.Matem.Mekhan.,Vol.47,No.1,pp.21-26,1983

$$m(t) = \int_U m(t, u)\, du \tag{1.6}$$

Let $H^n = \{(g_k(u))\}$ be a linear space of vectors each of whose coordinates is an element of space H (of generalized densities). The following are examples of linear operators acting in H^n:

$$Ag = \Big(\sum_{j=1}^{n} a_{kj} g_j\Big), \quad B(u) g = \Big(\sum_{j=1}^{n} b_{kj} g_j\Big), \quad L^* g = (L^* g_k)$$

It is seen that operators A and L^* commute on H^n.

By H_q we denote a uni-dimentional eigenvalue subspace in space H, specified by the function $q(u)$: $H_q = \{\gamma q(u)\}$. Obviously, $L^* H_q = 0$. Let H' be the image (or its closure) of operator L^* in H. We represent space H as the direct sum $H = H_q \dotplus H'$. Analogously, we represent $H^n = V_q \dotplus V'$, where

$$V_q = \{g(u)\} = \left\{ \begin{Vmatrix} c_1 q(u) \\ \vdots \\ c_n q(u) \end{Vmatrix} \right\} = \left\{ \begin{Vmatrix} c_1 \\ \vdots \\ c_n \end{Vmatrix} q(u) \right\} = \{C_0 q(u)\}$$

$$V' = \{g(u)\} = \left\{ \begin{Vmatrix} g_1(u) \\ \vdots \\ g_n(u) \end{Vmatrix} \right\}, \quad g_k(u) \in H', \quad g(u) \in V_q \Rightarrow$$

$$Ag \in V_q, \quad g(u) \in V' \Rightarrow Ag \in V', \quad L^* g \in V'$$

Thus, the solution $m(t, u)$ of system (1.5) is representable as

$$m(t, u) = m_0(t) q(u) + m'(t, u), \quad m(t, u) = \begin{Vmatrix} m_0(t) \\ m'(t, u) \end{Vmatrix}$$

$$m_0(t) q(u) \in V_q, \quad m'(t, u) \in V'$$

Let S_{01} and S_{10} be two linear operators acting, respectively, from the subspace V' into the subspace V_q and vice versa: $S_{01} V' \subseteq V_q$, $S_{10} V_q \subseteq V'$. We consider the vector

$$l(t, u) = \begin{Vmatrix} l_0(t) \\ l'(t, u) \end{Vmatrix}, \quad l_0(t) q(u) \in V_q, \quad l'(t, u) \in V'$$

connected with the vector $m(t, u)$ by the formulas

$$\begin{Vmatrix} m_0(t) \\ m'(t, u) \end{Vmatrix} = \begin{Vmatrix} E_0 & S_{01} \\ S_{10} & E' \end{Vmatrix} \begin{Vmatrix} l_0(t) \\ l'(t, u) \end{Vmatrix} = Sl(t, u)$$

where E_0 and E' are identity operators in subspaces V_q and V', respectively. The operator $B(u)$ $m(t, u)$ in a similar block-matrix form is written as

$$B(u) m(t, u) = \begin{Vmatrix} \Phi_{00} & \Phi_{01} \\ \Phi_{10} & \Phi' \end{Vmatrix} \begin{Vmatrix} m_0(t) \\ m'(t, u) \end{Vmatrix}$$

where Φ_{ij}, Φ' are linear operators acting inside the subspaces V_q and V', respectively, Φ_{00}, Φ', from subspace V' into subspace V_q, Φ_{01}, and vice versa, Φ_{10}. Substituting the expansions presented into Eq.(1.5), for $l(t, u)$ we obtain the equation

$$\frac{\partial}{\partial t} \begin{Vmatrix} l_0(t) \\ l'(t, u) \end{Vmatrix} = \begin{Vmatrix} Z_{00} & Z_{01} \\ Z_{10} & Z' \end{Vmatrix} \begin{Vmatrix} l_0(t) \\ l'(t, u) \end{Vmatrix}$$

where Z_{ij}, Z' are linear operators acting inside the subspaces V_q, V' and between them and having the form

$$\begin{aligned}
Z_{00} &= \Delta_0^{-1} (W_{00} + \mu \Phi_{01} S_{10}), \quad Z_{01} = \Delta_0^{-1} (W_{00} S_{01} + \mu \Phi_{01}) \\
Z_{10} &= \Delta_1^{-1} (W' S_{10} + \mu \Phi_{10} - S_{10} A - \mu S_{10} \Phi_{00}) \\
Z' &= \Delta_1^{-1} (W' + \mu \Phi_{10} S_{01} - S_{10} (A + \mu \Phi_{00}) S_{01}) \\
\Delta_0 &= E_0 - S_{01} S_{10}, \quad \Delta_1 = E' - S_{10} S_{01} \\
W_{00} &= A + \mu \Phi_{00} - \mu S_{01} \Phi_{10} + S_{01} (A + L^*) + \mu S_{01} \Phi' \\
W' &= A + L^* + \mu \Phi' - \mu S_{10} \Phi_{01}
\end{aligned}$$

Let us require that $Z_{01} = 0$, $Z_{10} = 0$. To be specific we investigate the first of these two equalities. We have the equation

$$A S_{01} - S_{01} (A + L^*) = \mu \Psi (S_{01}) \tag{1.7}$$

The solution of the inhomogeneous Eq. (1.7) is representable as

$$S_{01} = \mu \int_0^\infty \exp(-At)\, \Psi \exp(At)\, P_t'\, dt \tag{1.8}$$

where P_t' is the semigroup of bounded operators on V', defined by the equation $\partial\varphi'/\partial t = L^*\varphi'$. But on V' the semigroup P_t' admits of the estimate $\| P_t' \| \leqslant R \exp(-\lambda t)$. From the condition of separability of spectrum (1.3) follows the absolute convergence of integral (1.8) for any value of Ψ. Furthermore, the estimate $\| S_{01} \| \leqslant \text{const} \| \mu \Psi \|$ is valid. Equation (1.7) can be solved by the method of successive approximations by the formulas

$$S_{01}^0 = 0,\ AS_{01}^k - S_{01}^k (A + L^*) = \mu\Psi (S_{01}^{k-1})$$

If $\| S_{01}^k \| \leqslant \varkappa = \text{const}$ for all k, then

$$\| S_{01}^k - S_{01}^{k-1} \| \leqslant \mu\varkappa_1 \| S_{01}^{k-1} - S_{01}^{k-2} \|$$

with some constant $\varkappa_1$. This signifies the convergence of the successive approximations for sufficiently small μ. Thus

$$S_{01} = \lim_{k\to\infty} S_{01}^k, \quad S_{10} = \lim_{k\to\infty} S_{10}^k$$

Moreover, the estimates

$$\| S_{01} \| \leqslant |\mu| c_1, \quad \| S_{10} \| \leqslant |\mu| c_1$$

with some constant c_1 are valid. Allowing for

$$S_{01} (A + L^*) + \mu S_{01}\Phi' = AS_{01} + \mu\Phi_{00}S_{01} + \mu\Phi_{01} - \mu S_{01}\Phi_{10}S_{01}$$

and substituting this into the expression for Z_{00}, we have

$$Z_{00} = \Delta_0^{-1} (A + \mu\Phi_{00} - \mu S_{01}\Phi_{10})\, \Delta_0$$

Without loss of generality we can take it that all the eigenvalues $\{\alpha_k\}$ of matrix A lie in the strip $\delta < \operatorname{Re} \alpha_k < \delta + c,\ \delta > 0$. This can be achived by a suitable choice of the appropriate constant α and by the substitution $X = Y\exp(\alpha t)$ in Eq. (1.2). On V' the operator $\exp(At)\, P_t'$ admits of the estimate $\| \exp(At)\, P_t' \| \leqslant \text{const} \exp(-\delta t)$. For sufficiently small μ we can find constants $\delta(\mu) > 0,\ c(\mu) > 0$ such that the spectrum $\{\alpha_k(\mu),\ k = 1, \ldots, n\}$ of matrix Z_{00} lies in the strip $\delta(\mu) < \operatorname{Re} \alpha_k(\mu) < \delta(\mu) + c(\mu)$, while the solution of the equation $\partial l'/\partial t = Z'l'$ admits of the estimate $\| l'(t, \mu) \| \leqslant \text{const} \exp(-\delta(\mu)\, t)$. We make the reverse substitution. Since

$$\int_U m'(t, u)\, du = 0, \quad \int_U q(u)\, du = 1 \tag{1.9}$$

for $m_0(t) = m(t)$ we have

$$m_0(t)q = \exp(Kt)\, l_0(0)\, q + r(t)\, q, \quad K = Z_{00}, \quad r(t)\, q(u) = S_{01}\, l'(t, u)$$

where $r(t)$ is some n-dimensional vector admitting of the estimate

$$\| r(t) \| \leqslant \text{const} \exp(-\delta(\mu)\, t)$$

and which, obviously, always can be represented in the form $r(t) = R(t)\, l_0(0)$ with some variable matrix $R(t)$. We have the explicit expressions

$$l_0(0)\, q(u) = \Delta_0^{-1} (m_0(0)q - S_{01}\, m'(0, u))$$
$$l'(0, u) = \Delta_1^{-1} (-S_{10}m_0(0)q + m'(0, u))$$

whence it follows that the vector $m_0(0)$ can be represented as

$$l_0(0) = (E_0 + \mu C_1(\mu))\, m_0(0) = Cm_0(0)$$

with some matrix $C_1(\mu)$ analytically dependent on μ. Then

$$m_0(t)q = \exp(Kt)\, (C + o)\, m_0(0)\, q, \ \operatorname{Det} C \neq 0 \tag{1.10}$$

Integrating (1.10) with respect to u with due regard to (1.9), we obtain the theorem's assertion.

Note. The matrix K depends analytically on parameter μ in some neighborhood of zero.

The system of linear differential equations with constant coefficients, $X^\cdot = KX$ with a matrix K or any other similar to it, is called the limit system for system (1.2) in the sense of first moments. A convenient representation of matrix K in the form

$$K = A + \mu\Phi_{00} - \mu S_{01}\Phi_{10}$$

follows directly from the theorem's proof. The eigenvalues of matrix K are, in general, complex and their real parts are the Liapunov exponents of the first moments of the solutions of system (1.2). In particular, if $m(0) \neq 0$, then the number of rigorous Liapunov exponents does not exceed the dimension of system (1.2). The following statement can be proved analogously.

Corollary. Let the eigenvalues $\{\alpha_j\}$ of matrix A admit of the estimates

$$\operatorname{Re}(\alpha_i - \alpha_j) < c/2, \quad c = \text{const} < \lambda$$

Then the matrix $M(t) = EXX^*$ of second moments of the solutions of system (1.2) admists, for sufficiently small μ, of the representation

$$M(t) = \exp(K_0 t)(C_2 + o)\, M(0)$$

where K_2 and C_2 are constant linear operators in the space of nth-order symmetric square matrices and o is an infinitesimal operator as $t \to \infty$.

Note. Using the techniques of working with multidimensional matrices /2/, we can state and prove corresponding assertions for the higher-order moments.

2. Let us show the importance of the smallness of parameter μ to the proof of representation (1.4). Consider the equation

$$\chi^{\cdot\cdot} - (1 + \mu\xi_t)\, x = 0 \tag{2.1}$$

with a random coefficient ξ_t which is a uniform Markov process with two constants $\{\pm 1\}$ with the infinitesimal matrix

$$Q = \begin{Vmatrix} -\alpha & \alpha \\ \alpha & -\alpha \end{Vmatrix}$$

The system of moment Eqs.(1.5) with be a system of homogeneous linear differential equations with constant fourth-order coefficients and have the form

$$\frac{d}{dt}\begin{Vmatrix} m_1(t,-1) \\ m_2(t,-1) \\ m_1(t,1) \\ m_2(t,1) \end{Vmatrix} = \begin{Vmatrix} -\alpha & 1-\mu & \alpha & 0 \\ 1 & -\alpha & 0 & \alpha \\ \alpha & 0 & -\alpha & 1+\mu \\ 0 & \alpha & 1 & -\alpha \end{Vmatrix} \begin{Vmatrix} m_1(t,-1) \\ m_2(t,-1) \\ m_1(t,1) \\ m_2(t,1) \end{Vmatrix}$$

Its characteristic polynomials has four roots

$$z_{1,2} = -\alpha + (\alpha^2 + 1 \pm \Delta)^{1/2}, \quad z_{3,4} = -\alpha - (\alpha^2 + 1 \pm \Delta)^{1/2}, \quad \Delta = (4\alpha^2 + \mu^2)^{1/2}$$

The roots z_1 and z_4 are always real. Roots $z_{2,3}$ can be both real as well as complex, and, always $z_1 > \operatorname{Re} z_{2,3}, \operatorname{Re} z_{2,3} > z_4$. The roots $z_{2,3}$ have nonzero imaginary parts under the condition $\mu^2 > (\lambda^2 - 1)^2$. Let $C_k = (c_i^{k}), (i, k = 1, \ldots, 4)$ be the eigenvectors of the system's matrix, corresponding to the simple eigenvalues z_k. The system's general solution is written as

$$(m_i(t, j)) = \sum_{k=1}^{4} C_k \exp(z_k t), \quad j = \pm 1$$

Consider the second-order vector

$$m(t) = \begin{Vmatrix} m_1(t) \\ m_2(t) \end{Vmatrix} = \begin{Vmatrix} m_1(t,-1) + m_1(t,1) \\ m_2(t,-1) + m_2(t,1) \end{Vmatrix}$$

In space E^4 we consider the subspace E' of dimension 2, defined by the conditions

$$E' = \{(x_1, x_2, x_3, x_4)^* : x_1 + x_3 = 0,\ x_2 + x_4 = 0\}$$

If $X \in E'$, then X cannot be an eigenvector of the system's matrix. Indeed, otherwise this would be equivalent to the simultaneous fulfillment of the two matrix equalities

$$\begin{Vmatrix} 0 & 1 \mp \mu \\ 1 & 0 \end{Vmatrix} \begin{Vmatrix} x_1 \\ x_2 \end{Vmatrix} - (2\alpha + z) \begin{Vmatrix} x_1 \\ x_2 \end{Vmatrix} = 0$$

which is impossible when $\mu = 0$. This signifies that the vectors C_k cannot lie in subspace E'. We represent the real vector $m(t)$ as

$$m(t) = a_1 A_1 \exp(\gamma_1 t) + (a_2 A_2 \cos\beta_2 t + a_3 A_3 \sin\beta_2 t)\exp(\gamma_2 t) + a_4 A_4 \exp(\gamma_4 t) \tag{2.2}$$
$$\gamma_1 > \gamma_2 > \gamma_4, \ \beta_2 \neq 0, \ \gamma_j = \operatorname{Re} z_j, \ \beta_2 = \operatorname{Im} z_2$$

where A_1, A_4 are certain nonzero vectors, at least one of the vectors A_2 or A_3 is nonzero, a_k are arbitrary constants. Let vector $m(t)$ be representable in the form

$$m(t) = \exp(Kt)(C + o)\, m(0) = D_1 \exp(\gamma_1 t) + D_2 \exp(\gamma t) + o(\exp(\gamma t)) \tag{2.3}$$

where K is a constant second-order matrix. The real number γ_1 must be an eigenvalue of matrix K. The second eigenvalue γ must be real. The contradiction in representations (2.2) and (2.3) in the general case indicates the impossibility of representation (2.3).

3. Let us show the importance of condition (1.3) to the representation (1.4). We set $\alpha = 1$. We see that the rate coefficient λ of the convergence to the stationary distribution (1.1) equals zero. The estimate on the eigenvalues (1.3) equals zero as well. For every $\mu > 0$ the roots $z_{2,3}$ will have nonzero imaginary parts. Using the argument in Sect.2, we conclude that representation (2.3) is impossible.

The author thanks R.Z. Khas'minskii for discussions on the paper and for remarks.

REFERENCES

1. BYLOV B.F., VINOGRAD R.E., GROBMAN D.M. and NEMYTSKII V.V., Theory of Liapunov Exponents. Moscow, NAUKA, 1966.
2. KHRISANOV S.M., On Frisch's nonlinear equations. Ukr. Mat. Zh., Vol.32, No.1, 1980.

Translated by N.H.C.